Summary

In this session, I aimed to discuss how the HJ equation is connected to the maximum invariant set. I started with equation 13 from *Data-Driven Safety Filters*, which presents the cost function equation and its relation to the distance function. I was initially confused by the signed distance function, which requires a second variable, y, to measure the distance between sets. After Dr. Petrik's explanation, it became clearer to me what the role of this variable is, especially when he mentioned that we use the signed distance to ensure the gradient of this function exists on the boundary. In the next equation, the value function equation, I was trying to explain that we are looking for the supremum of the cost function, which corresponds to the infimum of the distance function. After that, I intended to explain the theorem, which is Proposition 4 in *A General Safety Framework*, and why we maximize the gradient of the value function with respect to the system dynamics, ensuring that this term is always positive.

In this meeting, there was a lot of discussion, and the most important one was about the transparency in the paper regarding how we transition from equation 13 to equation 20. Dr. Begum believes that this step is related to Nagumo’s theorem, where we are trying to maximize inequality. Additionally, we were confused about whether the equation is the HJ equation or the HJI equation. Dr. Yoon also raised a question about why we are maximizing the value function with respect to u and the cost function, and why we are again looking for another maximization problem where u maximizes the inequality. I’ve been thinking about it, and it seems that equation 14 is the key. First, we maximize the value function with respect to the input, then calculate the gradient, and again set up another maximization problem to find u that maximizes the dynamics.

From my understanding, the value function needs to have both an upper and lower bound. The reason for setting it up this way is to ensure the lower bound is zero and the upper bound is the supremum of the cost function. Therefore, regardless of whether it's the HJ or HJI equation, what I know is that the lower bound of these equations is zero, and the term that appears in both equations—namely, the gradient of the value function multiplied by the dynamics—must be positive.

I also agree that equation 20 is the HJI equation because the signed-distance function seems to be specifically designed for a differential game problem. As I mentioned earlier, in the set-theoretic method, the signed-distance function is also defined, and the variable w is introduced in the equation as an exogenous input (page 120 and page 124).

Dr. Begum suggested that for the next meeting, I focus on a specific example of Dubbin’s car and try to solve the problem, particularly equation 13, for this case. She also recommended that I bring the results, including the maximum invariant set for it.